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Fluctuations of Matrix Elements of Regular Functions of Gaussian Random Matrices

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Abstract We find the limit of the variance and prove the Central Limit Theorem (CLT) for the matrix elements $\varphi_{jk}(M)$, $j,k=1,\dots,n$ of a regular function φ of the Gaussian matrix M (GOE and GUE) as its size n tends to infinity. We show that unlike the linear eigenvalue statistics $\mathrm{Tr}\,\varphi(M)$, a traditional object of random matrix theory, whose variance is bounded as $n\to\infty$ and the CLT is valid for $\mathrm{Tr}\,\varphi(M)-\mathbf{E}\{\mathrm{Tr}\,\varphi(M)\}$, the variance of $\varphi_{jk}(M)$ is $O(1/n)$, and the CLT is valid for $\sqrt{n}(\varphi_{jk}(M)-\mathbf{E}\{\varphi_{jk}(M)\})$. This shows the role of eigenvectors in the forming of the asymptotic regime of various functions (statistics) of random matrices. Our proof is based on the use of the Fourier transform as a basic characteristic function, unlike the Stieltjes transform and moments, used in majority of works of the field. We also comment on the validity of analogous results for other random matrices.

Keywords Random matrices · Linear eigenvalue statistics · Central Limit Theorem

1 Introduction

Central Limit Theorems are an important element of the probabilistic approach. In particular, a number of the theorems is known in random matrix theory. However, an overwhelming majority of them deals with linear eigenvalue statistics, defined as

$$\mathcal{N}_n[\varphi]:=\sum_{l=1}^{n}\varphi(\lambda_l^{(n)})=\mathrm{Tr}\,\varphi(M), \tag{1}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is a test function and $\{\lambda_l^{(n)}\}_{l=1}^{n}$ are the eigenvalues of an $n\times n$ matrix M (real symmetric, hermitian, unitary, etc.). We refer the reader to recent papers [1, 3, 9–12, 16] and references therein.

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An important property of linear eigenvalue statistics (1) is that for many random matrices the variance of statistics is bounded as $n \to \infty$ if the corresponding test function has a bounded derivative. This has to be compared with the case of i.i.d. random variables $\{\xi_l^{(n)}\}_{l=1}^n$, where the variance of linear statistics is linear in n for any bounded φ (in fact, such that $\mathbf{E}\{|\varphi(\xi_l^{(n)})|^2\} < \infty$). This provides an intuitively simple mechanism of emergence of "regular" asymptotic behavior of normalized (by the square root of variance) and centered statistics as sums of large number of small terms

$$(n\,\mathbf{Var}\{\varphi(\xi_1^{(n)})\})^{-1/2}\varphi^\circ(\xi_l^{(n)}), \quad l = 1, \dots, n,$$

where $\varphi^\circ(\xi_l^{(n)}) = \varphi(\xi_l^{(n)}) - \mathbf{E}\{\varphi(\xi_l^{(n)})\}$.

On the other hand, in the case of linear eigenvalue statistics of random matrices the normalized and centered statistics $(\mathbf{Var}\{\mathcal{N}_n[\varphi]\})^{-1/2}\mathcal{N}_n^\circ[\varphi]$, where

$$\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\},$$

are the sum of large number of terms of order $O(1)$ as $n \to \infty$, and the emergence of a limiting law is the result of rather subtle cancellation effects that may [11] and may not [16] take place.

In this note we present a class of functions (statistics) of random matrices, whose large-n behavior is analogous to that of linear statistics of i.i.d. random variables, i.e., the variance of statistics is $O(n^{-1})$ as $n \to \infty$ and the normalized and centered statistics converge in distribution to the standard Gaussian random variable. The statistics are just the entries $\varphi_{jk}(M)$, $j, k = 1, \dots, n$, of the matrix $\varphi(M)$ in a certain basis (recall that $\mathcal{N}_n[\varphi]$ of (1) is the sum of all diagonal entries of $\varphi(M)$).

For the sake of simplicity of our presentation we confine ourselves to the "classical" ensembles of random matrices, the Gaussian Unitary Ensemble (GUE) and the Gaussian Orthogonal Ensemble (GOE), although certain results below are valid for broader classes of random matrices (see e.g. Remarks 1(1) and (2)).

Our result can be also viewed as an analog of the E. Borel theorem for matrix elements of orthogonal random matrices (see e.g. [8] and references therein).

The paper is organized as follows. In Sect. 2 we find the variance of matrix elements of the GUE and GOE matrices by using simple invariance argument, and in Sect. 3 we prove the corresponding Central Limit Theorems by using the integration by parts and a version of the Poincaré inequality for Gaussian random variables (a similar techniques were used in [7, 15] in a somewhat different context). Note that we write below the integrals without limits for the integrals over the real axis.

2 Variance of Matrix Elements

We consider real symmetric or hermitian $n \times n$ random matrices whose probability laws are

$$\frac{1}{Z_{n,\beta}} \exp\left(-\frac{n\beta}{4w^2}\operatorname{Tr} M^2\right) d_\beta M, \quad \beta = 1, 2, \tag{2}$$

where $\beta = 1$ for real symmetric matrices (GOE) and $\beta = 2$ for the hermitian matrices (GUE), $Z_{n,\beta}$ is a normalizing constant, and

$$d_1 M = \prod_{j=1}^n dM_{jj} \prod_{1 \le j < k \le n} dM_{jk}, \tag{3}$$

$$d_2M = \prod_{j=1}^{n} dM_{jj} \prod_{1\le j<k\le n} d\Re M_{jk} d\Im M_{jk}. \tag{4}$$

Theorem 1 *Consider the Gaussian Ensembles, defined by* (2)–(4). *Let* $\varphi_{1,2} : \mathbb{R} \to \mathbb{C}$ *be bounded differentiable functions with bounded derivatives. Then we have for any* $j = 1, \ldots, n$

$$\lim_{n\to\infty} n\,\mathbf{Cov}\{(\varphi_1(M))_{jj}, (\varphi_2(M))_{jj}\} = \frac{1}{\beta}\int_{-2w}^{2w}\int_{-2w}^{2w} \Delta\varphi_1 \Delta\varphi_2 \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2, \tag{5}$$

where $\Delta\varphi = \varphi(\lambda_1) - \varphi(\lambda_2)$, ρ_{sc} *is the density of the semicircle law*

$$\rho_{sc}(\lambda) = (2\pi w^2)^{-1}(4w^2 - \lambda^2)_+^{1/2}, \tag{6}$$

and $x_+ = \max\{0, x\}$.

Proof Consider first the GUE. It follows from its unitary invariance that for any $j = 1, \ldots, n$

$$\mathbf{E}\{\varphi_{jj}(M)\} = \mathbf{E}\{\varphi_{jj}(U^*MU)\} = \mathbf{E}\{(U^*\varphi(M)U)_{jj}\} = \sum_{k_1,k_2=1}^{n} \overline{U}_{k_1 j} U_{k_2 j}\, \mathbf{E}\{\varphi_{k_1 k_2}(M)\},$$

where U is an arbitrary unitary matrix. Integrating this over U with respect to the Haar measure $H_{U(n)}$ of $U(n)$, we obtain

$$\mathbf{E}\{\varphi_{jj}(M)\} = \mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi(M)\} = \mathbf{E}\{n^{-1}\mathcal{N}_n[\varphi]\} \tag{7}$$

in view of the relation

$$\int_{U(n)} \overline{U}_{k_1 j} U_{k_2 j} H_{U(n)}(dU) = n^{-1}\delta_{k_1 k_2} \tag{8}$$

that is just the orthogonality relation of representation theory. It can also be obtained by observing that for any $j = 1, \ldots, n$ $\{U_{kj}\}_{k=1}^{n}$ is the random vector, uniformly distributed over the n-dimensional complex unit sphere

$$\sum_{k=1}^{n} |U_{kj}|^2 = 1. \tag{9}$$

Furthermore

$$\mathbf{E}\{(\varphi_1(M))_{jj}(\varphi_2(M))_{jj}\} = \sum_{k_1,k_2,k_3,k_4=1}^{n} \overline{U}_{k_1 j} U_{k_2 j} \overline{U}_{k_3 j} U_{k_4 j}\, \mathbf{E}\{(\varphi_1(M))_{k_1 k_2}(\varphi_2(M))_{k_3 k_4}\}.$$

Viewing again $\{U_{kj}\}_{k=1}^{n}$ as the random vector uniformly distributed over (9), we obtain

$$\int_{U(n)} \overline{U}_{k_1 j} U_{k_2 j} \overline{U}_{k_3 j} U_{k_4 j} H_{U(n)}(dU) = (n(n+1))^{-1}(\delta_{k_1 k_2}\delta_{k_3 k_4} + \delta_{k_1 k_4}\delta_{k_2 k_3}). \tag{10}$$

This implies that

$$\mathbf{E}\{(\varphi_1(M))_{jj}(\varphi_2(M))_{jj}\} = \frac{\mathbf{E}\{\operatorname{Tr}\varphi_1(M)\operatorname{Tr}\varphi_2(M)\} + \mathbf{E}\{\operatorname{Tr}\varphi_1(M)\varphi_2(M)\}}{n(n+1)},$$

and in view of (7)

$$\begin{aligned} n\,\mathbf{Cov}\{(\varphi_1(M))_{jj},(\varphi_2(M))_{jj}\} \\ &= \frac{1}{n+1}\mathbf{Cov}\{\mathcal{N}_n[\varphi_1],\mathcal{N}_n[\varphi_2]\} \\ &\quad + \frac{n}{n+1}(\mathbf{E}\{n^{-1}\operatorname{Tr}\varphi_1(M)\varphi_2(M)\} - \mathbf{E}\{n^{-1}\operatorname{Tr}\varphi_1(M)\}\,\mathbf{E}\{n^{-1}\operatorname{Tr}\varphi_2(M)\}). \end{aligned} \tag{11}$$

It is well known in the random matrix theory (see e.g. [2, 13, 15] and references therein) that there exists a nonnegative function ρ_n such that for any bounded and continuous φ

$$\mathbf{E}\{n^{-1}\operatorname{Tr}\varphi(M)\} = \int \varphi(\lambda)\rho_n(\lambda)d\lambda, \tag{12}$$

and we have with probability 1

$$\lim_{n\to\infty} n^{-1}\operatorname{Tr}\varphi(M) = \int_{-2w}^{2w} \varphi(\lambda)\rho_{sc}(\lambda)d\lambda \tag{13}$$

with ρ_{sc} of (6). This and the bound

$$|n^{-1}\operatorname{Tr}\varphi(M)| \le \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)| \tag{14}$$

imply that the limit of the second term of (11) is the r.h.s. of (5):

$$\begin{aligned} &\int \varphi_1(\lambda)\varphi_2(\lambda)\rho_{sc}(\lambda)d\lambda - \int \varphi_1(\lambda)\rho_{sc}(\lambda)d\lambda \int \varphi_2(\lambda)\rho_{sc}(\lambda)d\lambda \\ &\quad = \frac{1}{2}\int (\varphi_1(\lambda)-\varphi_1(\mu))(\varphi_2(\lambda)-\varphi_2(\mu))\rho_{sc}(\lambda)\rho_{sc}(\mu)d\lambda d\mu. \end{aligned}$$

Let us show that the limit of the first term of (11) is zero. To this end we use the Poincaré inequality (see e.g. [4]) for independent Gaussian random variables $\{\xi_l\}_{l=1}^p$, $\mathbf{E}\{\xi_l\}=0$, $\mathbf{E}\{\xi_l^2\}=\sigma_l^2$ and $\Phi:\mathbb{R}^p\to\mathbb{C}$ with bounded first derivatives:

$$\mathbf{Var}\{\Phi(\xi_1,\dots,\xi_p)\} \le \sum_{l=1}^p \sigma_l^2\,\mathbf{E}\left\{\left|\frac{\partial}{\partial\xi_l}\Phi\right|^2\right\}. \tag{15}$$

Choosing here

$$\left\{\{M_{jj}\}_{j=1}^n,\ \{\Re M_{jk}\}_{1\le j<k\le n},\ \{\Im M_{jk}\}_{1\le j<k\le n}\right\}, \tag{16}$$

where (see (2) with $\beta=2$)

$$\mathbf{E}\{M_{jj}\} = \mathbf{E}\{\Re M_{jk}\} = \mathbf{E}\{\Im M_{jk}\} = 0, \tag{17}$$

$$\mathbf{E}\{M_{jj}^2\} = w^2n^{-1},\ \mathbf{E}\{(\Re M_{jk})^2\} = \mathbf{E}\{(\Im M_{jk})^2\} = w^2(2n)^{-1}, \tag{18}$$

as $\{\xi_l\}_{l=1}^p$, and $\mathrm{Tr}\,\varphi(M)$ as Φ, we obtain in view of the relation

$$\frac{\partial}{\partial M_{ab}}\mathrm{Tr}\,\varphi(M)=\varphi'_{ba}(M)$$

the following bounds

$$\mathbf{Var}\{\mathrm{Tr}\,\varphi(M)\}\le w^2\,\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi'(M)(\varphi'(M))^*\} \tag{19}$$

$$\le w^2(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|)^2 \tag{20}$$

(see [7, 15] for similar bounds). This implies that the first term on the r.h.s. is $O(n^{-1})$ as $n\to\infty$. We proved (5) for $\beta=2$.

The proof for GOE is similar. We have an exact analog of (8)

$$\int_{O(n)} O_{k_1j}O_{k_2j}H_{O(n)}(dO)=n^{-1}\delta_{k_1k_2},$$

an analog of (10)

$$\int_{O(n)} O_{k_1j}O_{k_2j}O_{k_3j}O_{k_4j}H_{O(n)}(dO)=\frac{\delta_{k_1k_2}\delta_{k_3k_4}+\delta_{k_1k_3}\delta_{k_2k_4}+\delta_{k_1k_4}\delta_{k_2k_3}}{n(n+2)},$$

and of (19)–(20):

$$\mathbf{Var}\{\mathrm{Tr}\,\varphi(M)\}\le 2w^2\,\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi'(M)(\varphi'(M))^*\}$$

$$\le 2w^2(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|)^2.$$

By the way, formula (11) and its analog for the GOE can be written in the compact form

$$\begin{aligned} &n\mathbf{Cov}\{(\varphi_1(M))_{jj},(\varphi_2(M))_{jj}\}\\ &\quad=\frac{\beta}{\beta n+2}\mathbf{Cov}\{\mathcal{N}_n[\varphi_1],\mathcal{N}_n[\varphi_2]\}\\ &\qquad+\frac{2n}{\beta n+2}(\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi_1(M)\varphi_2(M)\}-\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi_1(M)\}\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi_2(M)\}).\end{aligned}$$ □

Remark 1 (1) It can be shown by using the orthogonal polynomial technique (see e.g. [13]) that if $p_1^{(n)}=\rho_n$ (see (12)) and $p_2^{(n)}$ are the first and the second marginals of the joint probability density of eigenvalues, then there exists $C<\infty$ and $c>0$ such that

$$\rho_n(\lambda)\le Ce^{-cn\lambda^2},\qquad p_2^{(n)}(\lambda_1,\lambda_2)\le Ce^{-cn(\lambda_1^2+\lambda_2^2)},\qquad |\lambda|\ge(2+\delta)w,\quad\forall\delta>0. \tag{21}$$

This and a standard approximation argument allow us to extend the theorem to test functions that admit the bound

$$|\varphi(\lambda)|\le Ae^{a\lambda^2},\quad |\lambda|\ge(2+\delta)w, \tag{22}$$

for some $A<\infty$, $a>0$, and $\delta>0$, and have a bounded derivative for $|\lambda|\le(2+\varepsilon)w$. Moreover, in the case of GUE the same technique yields the asymptotic relation $\mathbf{Cov}\{\mathrm{Tr}\,\varphi_1(M), \mathrm{Tr}\,\varphi_2(M)\} = o(n)$ for continuous $\varphi_{1,2}$. Combining this with (21), we obtain the validity of the theorem for the test functions that are continuous on the interval $[-2w-\varepsilon, 2w+\varepsilon]$ and are exponentially bounded outside of this interval.

(2) Similar result can be proved for the off-diagonal entries $\varphi_{jk}(M)$, $j \neq k$. Consider, for instance, the GUE. Here $\mathbf{E}\{\varphi_{jk}(M)\} = 0$ by (8), and it can be shown that (cf. (11))

$$n\,\mathbf{Cov}\{(\varphi_1(M))_{jk}, (\varphi_2(M))_{jk}\}$$
$$= \frac{n^2}{n^2-1}(\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi_1(M)\varphi_2(M)\} - \mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi_1(M)\}\,\mathbf{E}\{n^{-1}\,\mathrm{Tr}\,\varphi_2(M)\})$$

(see e.g. [6], where the general 4th moment of entries of $U \in U(n)$ is given). This, (13), and (14) show that $n\,\mathbf{Cov}\{(\varphi_1(M))_{jk}, (\varphi_2(M))_{jk}\}$ for $j \neq k$ has the same limit as $n\,\mathbf{Cov}\{(\varphi_1(M))_{jj}, (\varphi_2(M))_{jj}\}$ in (5) but now for any bounded continuous $\varphi_{1,2}$ in view of (13), while in (5) $\varphi_{1,2}$ are bounded together with their first derivatives (see, however, the previous remark).

(3) The theorem can also be extended to the hermitian and real symmetric matrix models, defined by (2) in which M^2 is replaced by $V(M)$, where $V : \mathbb{R} \to \mathbb{R}$ is bounded below, continuous, and admits the bound $V(\lambda) \geq (2+\varepsilon)\log(|\lambda|+1)$ for sufficiently large λ. This can be done by using the results of [5, 17, 18].

We turn now to the Central Limit Theorem for $(\varphi(M))_{jj}$.

3 Central Limit Theorem for Matrix Elements

Theorem 2 *Consider the Gaussian Ensembles defined by* (2)–(4), *and denote for any bounded differentiable* $\varphi : \mathbb{R} \to \mathbb{R}$ *with bounded derivative*

$$\varphi^{\circ}_{jj}(M) = (\varphi(M))_{jj} - \mathbf{E}\{(\varphi(M))_{jj}\}. \tag{23}$$

Then for any $j = j_n \in [1, n]$ *the random variable* $\sqrt{n}\varphi^{\circ}(M)_{j_n j_n}$ *converges in distribution to the Gaussian random variable with zero mean and the variance* (*cf.* (5))

$$V_d[\varphi] = \frac{1}{\beta}\int_{-2w}^{2w}\int_{-2w}^{2w} |\varphi(\lambda_1) - \varphi(\lambda_2)|^2 \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2)d\lambda_1 d\lambda_2, \tag{24}$$

where ρ_{sc} *is the density* (6) *of the semicircle law.*

Proof We consider the technically simplest case of the GUE, given by (4) and (2) with $\beta = 2$. In view of the unitary invariance of the GUE (see (2) and (4)) we can confine ourselves to the case of $j = 1$ in (23) without loss of generality. By the continuity theorem for characteristic functions it suffices to show that if

$$Z_{1n}(x) = \mathbf{E}\left\{e^{ix\sqrt{n}\varphi^{\circ}_{11}(M)}\right\}, \quad x \in \mathbb{R}, \tag{25}$$

then for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} Z_{1n}(x) = Z_d(x), \tag{26}$$

where

$$Z_d(x) = \exp\left\{-x^2 V_d[\varphi]/2\right\}. \tag{27}$$

Assume first that φ admits the Fourier transform

$$\widehat{\varphi}(t) = \frac{1}{2\pi}\int e^{-i\lambda t}\varphi(\lambda)d\lambda,$$

such that

$$\int (1+|t|)^2|\widehat{\varphi}(t)|dt < \infty. \tag{28}$$

Since $Z_{1n}(0) = 1$, we can write the relation

$$Z_{1n}(x) = 1 + \int_0^x Z'_{1n}(y)dy, \tag{29}$$

showing that it suffices to prove that the sequences $\{Z'_{1n}\}$ is bounded on any finite interval and that for any converging subsequences $\{Z_{1n_l}\}_{l\geq 1}$ and $\{Z'_{1n_l}\}_{l\geq 1}$ we have

$$\lim_{l\to\infty} Z'_{1n_l}(x) = -xV_d[\varphi]Z(x), \quad Z(x) = \lim_{l\to\infty} Z_{1n_l}(x). \tag{30}$$

Indeed, if yes, we obtain from (29) the equation

$$Z(x) = 1 - V_d[\varphi]\int_0^x yZ(y)dy \tag{31}$$

whose unique solution is (27).

Note first that according to (25)

$$Z'_{1n}(x) = i\mathbf{E}\left\{\sqrt{n}\varphi^{\circ}_{11}(M)e^{ix\sqrt{n}\varphi^{\circ}_{11}(M)}\right\}. \tag{32}$$

Then, the Fourier inversion formula

$$\varphi(\lambda) = \int e^{i\lambda t}\widehat{\varphi}(t)dt, \tag{33}$$

and the spectral theorem for hermitian matrices yield

$$\varphi^{\circ}_{11}(M) = \int \widehat{\varphi}(t)U^{\circ}_{11}(t)dt, \tag{34}$$

where

$$U^{\circ}_{11}(t) = U_{11}(t) - \mathbf{E}\{U_{11}(t)\}, \qquad U(t) = e^{itM} = \{U_{jk}\}^n_{j,k=1}.$$

This and (32) imply

$$Z'_{1n}(x) = i\int \widehat{\varphi}(t)Y_{1n}(x,t)dt, \tag{35}$$

where

$$Y_{1n}(x,t) = \sqrt{n}\,\mathbf{E}\{U_{11}^{\circ}(t)e_{1n}(x)\}, \quad e_{1n}(x) = e^{ix\sqrt{n}\varphi_{11}^{\circ}(M)}. \tag{36}$$

Since $\overline{Y_{1n}(x,t)} = Y_{1n}(-x,-t)$, it suffices to consider Y_{1n} on the domain $\{t \geq 0,\ x \in \mathbb{R}\}$.

We will prove that the sequence $\{Y_{1n}\}$ is bounded and equicontinuous on any finite set of $\{t \geq 0,\ x \in \mathbb{R}\}$ and that its every uniformly converging on the set subsequence has the same limit Y_d, leading to (30). This proves the assertion of the theorem under condition (28). Indeed, let $\{Z_{1n_l}\}_{l\geq 1}$ be the subsequence, converging to $\widetilde{Z}_d \neq Z_d$. Consider the corresponding subsequence $\{Y_{1n_l}\}_{l\geq 1}$. It contains a uniformly converging sub-subsequence, whose limit is again Y_d and this forces the corresponding subsequence of $\{Z_{1n_l}\}_{l\geq 1}$ to converge to Z_d, a contradiction.

We will need the Duhamel formula

$$e^{itM_2} = e^{itM_1} + i\int_0^t e^{i(t-s)M_2}(M_2 - M_1)e^{isM_1}ds, \tag{37}$$

valid for any $n \times n$ matrices $M_{1,2}$. The formula implies that

$$\frac{\partial}{\partial M_{ab}}U_{11}(t) = i\int_0^t U_{1a}(t-s)U_{b1}(s)ds, \tag{38}$$

in particular, that (see (9))

$$\left|\frac{\partial}{\partial M_{ab}}U_{11}(t)\right| \leq t,$$

where we used the bound $|U_{11}(t)| \leq 1,\ \forall t \in \mathbb{R}$ (cf. (9)).

It follows then from (15) with $\Phi = U_{11}$, (16)–(18), and (38) that (cf. (20))

$$\mathbf{Var}\{U_{11}(t)\} \leq w^2t^2/n, \tag{39}$$

and we obtain from (36) and the Schwarz inequality that

$$|Y_{1n}(x,t)| \leq wt. \tag{40}$$

Thus, Y_{1n} is bounded on any bounded set of $\mathbb{R}^2$.

Next, it follows from (34) that

$$\frac{\partial}{\partial x}Y_{1n}(x,t) = in\int \widehat{\varphi}(s)\mathbf{E}\{U_{11}^{\circ}(s)U_{11}^{\circ}(t)e_{1n}(x)\}ds,$$

and since (36), (39), and the Schwarz inequality imply the bound

$$|\mathbf{E}\{U_{11}^{\circ}(s)U_{11}^{\circ}(t)e_{1n}(x)\}| \leq \mathbf{Var}^{1/2}\{U_{11}(s)\}\,\mathbf{Var}^{1/2}\{U_{11}(t)\} \leq w^2st/n,$$

we obtain in view of (28) that the sequence $\{\frac{\partial}{\partial x}Y_{1n}\}$ is bounded uniformly in t and x.

We have also

$$\frac{\partial}{\partial t}Y_{1n}(x,t) = i\sqrt{n}\,\mathbf{E}\{(MU)_{11}^{\circ}(t)e_{1n}(x)\}, \tag{41}$$

and using the Schwarz inequality, (15) with $\Phi = (MU)_{11}$, (16)–(18), and (38), we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial t}Y_{1n}(x,t)\right|^2 &\le n\,\mathbf{Var}\{(MU)_{11}\}\\ &\le 2w^2\left(1+\int_0^t ds_1\int_0^t ds_2\mathbf{E}\{(MU(s_1-s_2)M)_{11}U_{11}(s_2-s_1)\}\right)\\ &\le 2w^2(1+w^2t^2).\end{aligned}$$

We conclude that the sequence $\{Y_{1n}\}$ is equicontinuous on any bounded set of $\mathbb{R}^2$.

We will prove now that any uniformly convergent subsequence of $\{Y_{1n}\}$ has the same limit, leading to (30)–(31), hence to (26)–(27).

We use the formula

$$\mathbf{E}\{\zeta\Phi(\zeta,\overline{\zeta})\} = \mathbf{E}\{|\zeta|^2\}\mathbf{E}\left\{\frac{\partial}{\partial\overline{\zeta}}\Phi(\zeta,\overline{\zeta})\right\}, \tag{42}$$

that is valid for any complex Gaussian variable $\zeta = \xi + i\eta$ with independent ξ and η and a differentiable $\Phi : \mathbb{R}^2 \to \mathbb{C}$, and can be easily proved by integration by parts. By using the formula and the relations

$$U_{11}(t) = 1 + i\int_0^t (MU(s))_{11}ds$$

and

$$\frac{\partial\varphi_{11}}{\partial M_{kj}} = i\int\widehat{\varphi}(t)dt\int_0^t U_{1k}(s)U_{j1}(t-s)ds$$

we obtain

$$\begin{aligned}Y_{1n}(x,t) &= i\sqrt{n}\int_0^t\sum_{k=1}^n\mathbf{E}\{M_{1k}U_{k1}(s)e_{1n}^\circ(x)\}ds\\ &= -w^2\sqrt{n}\int_0^t ds\int_0^s\mathbf{E}\{v_n(s-s_1)U_{11}(s_1)e_{1n}^\circ(x)\}ds_1\\ &\quad - iw^2x\int_0^t ds\int\widehat{\varphi}(t_1)dt_1\int_0^{t_1}\mathbf{E}\{U_{11}(s+s_1)U_{11}(t_1-s_1)e_{1n}(x)\}ds_1,\end{aligned}$$

where

$$v_n(t) = n^{-1}\,\mathrm{Tr}\,U(t). \tag{43}$$

Introducing the quantities

$$v_n^\circ = v_n - \overline{v}_n, \quad \overline{v}_n = \mathbf{E}\{v_n\}, \tag{44}$$

we rewrite the above relation as

$$\begin{aligned}&Y_{1n}(x,t) + w^2\int_0^t ds\int_0^s\overline{v}(s-s_1)Y_{1n}(x,s_1)\\ &\quad = -iw^2xZ_{1n}(x)\int_0^t ds\int\widehat{\varphi}(t_1)\Phi_n(s,t_1)dt_1 + r_n(x,t),\end{aligned}$$

where

$$\Phi_n(s,t_1)=\int_0^{t_1}\mathbf{E}\{U_{11}(s+s_1)U_{11}(t_1-s_1)\}ds_1$$

and

$$\begin{aligned}r_n(x,t)=&-w^2\sqrt{n}\int_0^t ds\int_0^s \mathbf{E}\{v_n^\circ(s-s_1)U_{11}(s_1)e_{1n}^\circ\}ds_1\\&-iw^2x\int_0^t ds\int\widehat{\varphi}(t_1)dt_1\int_0^{t_1}\mathbf{E}\{(U_{11}(s+s_1)U_{11}(t_1-s_1))^\circ e_{1n}\}ds_1.\end{aligned}$$

The integrand in the first term of the r.h.s. can be estimated by using (20) with $\varphi=n^{-1}e^{it\lambda}$:

$$|\mathbf{E}\{v_n^\circ(s-s_1)U_{11}(s_1)e_{1n}^\circ\}|\le\mathbf{E}\{|v_n^\circ(s-s_1)|\}\le\mathbf{Var}^{1/2}\{v_n(s-s_1)\}\le w(s-s_1)/n,$$

hence the first term of the r.h.s. admits the bound $w^3t^3/6\sqrt{n}$.

Furthermore, by using the inequalities $|U_{11}(t)|\le 1,\ \forall t\in\mathbb{R}$ and

$$\mathbf{E}\{|(\xi_1\xi_2)^\circ|\}\le 2c\,\mathbf{E}\{|\xi_1{}^\circ|\}+2c\,\mathbf{E}\{|\xi_2{}^\circ|\},\quad |\xi_{1,2}|\le c,$$

where $\xi_{1,2}^\circ=\xi_{1,2}-\mathbf{E}\{\xi_{1,2}\}$, and $\xi_{1,2}$ are any random variables, we obtain in view of (39) and (28) that

$$|r_n(x,t)|\le C(x,t)n^{-1/2},$$

where $C(x,t)$ is n-independent and polynomial in x and t.

This implies that the limit Y_d of every uniformly converging subsequence of $\{Y_{1n}\}$ satisfies the equation

$$\begin{aligned}Y_d(x,t)+w^2\int_0^t ds\int_0^s v(s-s_1)Y_d(x,s_1)ds_1\\=-iw^2xZ_d(x)\int_0^t ds\int\widehat{\varphi}(t_1)\Phi(s,t_1)dt_1,\end{aligned}\tag{45}$$

where

$$v(t)=\int_{-2w}^{2w}e^{it\lambda}\rho_{sc}(\lambda)d\lambda,\tag{46}$$

(see (13), (43), and (44)), Z_d is the limit of the corresponding converging subsequence of $\{Z_{1n}\}$, and

$$\Phi(s,t)=\int_0^t v(s+s_1)v(t-s_1)ds_1.\tag{47}$$

To solve (45) we use the generalized Fourier transform [19], in fact the $\pi/2$ rotated Laplace transform with respect to t:

$$\begin{aligned}\widetilde{Y}_d(x,z)&=-i\int_0^\infty e^{-itz}Y_d(x,t)dt,\qquad \Im z\le-\varepsilon<0,\\ Y_d(x,t)&=\frac{i}{2\pi}\int_L e^{itz}\widetilde{Y}_d(x,z)dz,\quad L=(-\infty-i\varepsilon,\infty-i\varepsilon).\end{aligned}$$

We obtain from (45) that

$$Y_d(x,t) = i w^2 x Z_d(x) \int_0^t T(t-s) \int \widehat{\varphi}(t_1)\Phi(t_1,s)dt_1, \tag{48}$$

where

$$T(t) = \frac{i}{2\pi} \int_L e^{itz} \frac{dz}{z + w^2 \widetilde{v}(z)},$$

and in view of (46) and (6)

$$\widetilde{v}(z) = -i \int_0^\infty e^{-itz} v(t)dt = (2w^2)^{-1}(\sqrt{z^2 - 4w^2} - z),$$

with the branch that is determined by the asymptotic $\sqrt{z^2 - 4w^2} = z + O(z^{-1})$, $z \to \infty$. This imply that

$$T(t) = \frac{1}{2\pi i} \int_L e^{itz} \widetilde{v}(z)dz = -v(t),$$

and then (48) and a little algebra yield

$$\begin{aligned} Y_d(x,t) &= i w^2 x Z_d(x) \\ &\times \int_{[-2w,2w]^3} \frac{e^{i\lambda_1 t} - e^{i\lambda_2 t}}{\lambda_1 - \lambda_2} \frac{\varphi(\lambda_2) - \varphi(\lambda_3)}{\lambda_2 - \lambda_3} \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2)\rho_{sc}(\lambda_3)d\lambda_1 d\lambda_2 d\lambda_3. \end{aligned}$$

Besides, a version of the above argument implies that a bounded solution of (45) is unique.

We obtain then in view of (35)

$$\begin{aligned} \lim_{l\to\infty} Z'_{1n_l}(x) = w^2 x Z_d(x) \int_{[-2w,2w]^3} &\frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2} \frac{\varphi(\lambda_2) - \varphi(\lambda_3)}{\lambda_2 - \lambda_3} \\ &\times \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2)\rho_{sc}(\lambda_3)d\lambda_1 d\lambda_2 d\lambda_3. \end{aligned} \tag{49}$$

Writing the numerator in the integral as

$$\varphi(\lambda_1)\varphi(\lambda_2) - \varphi(\lambda_1)\varphi(\lambda_3) - \varphi^2(\lambda_2) + \varphi(\lambda_2)\varphi(\lambda_3), \tag{50}$$

we observe that there is at least one integration which does not involve φ's. This and the relation

$$\int \frac{\rho_{sc}(\mu)d\mu}{\mu - \lambda} = -\lambda/2w^2$$

allow us to deduce (24) from (49). A simple way to perform the corresponding calculations is to write the r.h.s. of (49) as the limit as $\varepsilon \to 0$ of the same expression in which λ_3 is replaced by $\lambda_3 + i\varepsilon$. One can also use the Poincaré–Bertrand formula [14] to deal with double singular integrals, appearing after plugging (50) in (49).

This proves the assertion of the theorem under condition (28).

The case of bounded C^1 test functions with bounded derivative can be obtained via a standard approximation procedure. Indeed, for any bounded C^1 function φ with bounded derivative there exists a sequence $\{\varphi_k\}$ of functions, satisfying (28) and such that we have for some $A \geq 2w$

$$\lim_{k\to\infty} \sup_{|\lambda|\leq A} |\varphi(\lambda) - \varphi_k(\lambda)| = 0. \tag{51}$$

Denote for the moment the characteristic functions of (25) and (27) as $Z_{1n}[\varphi]$ and $Z_d[\varphi]$, to make explicit their dependence on test function. We have then

$$|Z_n[\varphi]-Z_d[\varphi]|\le|Z_n[\varphi]-Z_n[\varphi_k]|+|Z_n[\varphi_k]-Z_d[\varphi_k]|+|Z[\varphi_k]-Z_d[\varphi]| \\ =T_{nk}^{(1)}+T_{nk}^{(2)}+T_k^{(3)}. \quad (52)$$

The second term $T_{nk}^{(2)}$ of the r.h.s. vanishes after the limit $n\to\infty$ in view of the above proof, since φ_k satisfies (28). For the first term we have from (25) and the Schwarz inequality that

$$|T_{nk}^{(1)}|\le|x|(n\,\mathbf{Var}\{(\psi_k(M))_{11}\})^{1/2},\quad \psi_k=\varphi-\varphi_k, \quad (53)$$

and then Theorem 1 with $\varphi_1=\varphi_2=\psi_k$ implies that

$$\limsup_{n\to\infty}|T_{nk}^{(1)}|\le|x|(V_d[\psi_k])^{1/2}.$$

At last we have from (27)

$$|T_k^{(3)}|\le x^2|V_d[\varphi]-V_d[\varphi_k]|/2,$$

and taking into account the continuity of V_d of (24) with respect to the C convergence on any interval $|\lambda|\le A$, $A>2w$, we find that the first and the third term of (52) vanish after the limit $k\to\infty$. This proves the Central Limit Theorem for bounded C^1 test functions with bounded derivative. □

Remark 2 (1) The theorem is also valid for test functions that are exponentially bounded (see (22)), and have a bounded derivative on the interval $[-(2+\varepsilon)w,(2+\varepsilon)w]$, $\varepsilon>0$. Indeed, if φ is a described function, we denote φ_1 a C^1 function, coinciding with φ for $|\lambda|\le(2+\varepsilon/2)w$, equal zero for $|\lambda|\ge(2+\varepsilon)w$ and such that if $\psi=|\varphi-\varphi_1|$, then $\psi\le|\varphi|$. We have (cf. (53)):

$$|Z_{1n}[\varphi]-Z_{1n}[\varphi_1]|\le|x|n^{1/2}\,\mathbf{E}\{\psi^\circ(M)_{11}\} \\ \le 2|x|n^{1/2}\int_{|\lambda|\ge(2+\varepsilon/2)w}|\varphi(\lambda)|\rho_n(\lambda)d\lambda,$$

and in view of (22) and (21) with $\delta=\varepsilon/2$ the expression on the r.h.s. is exponentially small as $n\to\infty$. This reduces the proof of the theorem for exponentially bounded functions to that for C^1 functions with support in $|\lambda|\le(2+\varepsilon)$, for which Theorem 2 is applicable.

(2) Similar, in fact simpler, argument proves the Central Limit Theorem for the off-diagonal matrix elements $\varphi_{jk}(M)$, $j\ne k$. The corresponding variance coincides with (24), but the CLT is valid now for any bounded continuous φ (see Remark 1(2)).

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